BOARD OF STUDIES
NEW SOUTH WALES

2011
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int x \ln x \, dx$. **2**

(b) Evaluate $\displaystyle\int_0^3 x\sqrt{x+1}\, dx$. **3**

(c) (i) Find real numbers a, b and c such that **2**

$$\frac{1}{x^2(x-1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}.$$

(ii) Hence, find $\displaystyle\int \frac{1}{x^2(x-1)}\, dx$. **2**

(d) Find $\displaystyle\int \cos^3\theta \, d\theta$. **3**

(e) Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{5-2t+t^2}\, dt$. **3**

Question 2 (15 marks) Use a SEPARATE writing booklet.

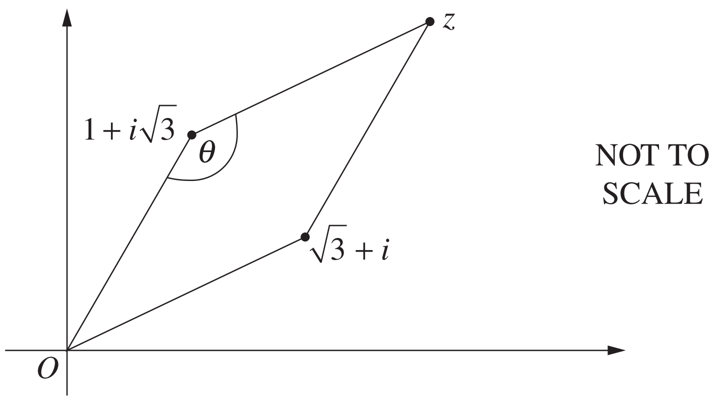
(a) Let $w = 2 - 3i$ and $z = 3 + 4i$.

(i) Find $\overline{w} + z$. **1**

(ii) Find $|w|$. **1**

(iii) Express $\dfrac{w}{z}$ in the form $a + ib$, where a and b are real numbers. **2**

(b) On the Argand diagram, the complex numbers 0, $1 + i\sqrt{3}$, $\sqrt{3} + i$ and z form a rhombus.



(i) Find z in the form $a + ib$, where a and b are real numbers. **1**

(ii) An interior angle, θ, of the rhombus is marked on the diagram. **2**

Find the value of θ.

(c) Find, in modulus-argument form, all solutions of $z^3 = 8$. **2**

(d) (i) Use the binomial theorem to expand $(\cos\theta + i\sin\theta)^3$. **1**

(ii) Use de Moivre's theorem and your result from part (i) to prove that **3**

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta .$$

(iii) Hence, or otherwise, find the smallest positive solution of **2**

$$4\cos^3\theta - 3\cos\theta = 1.$$

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Draw a one-third page sketch of the graph $y = \sin\frac{\pi}{2}x$ for $0 < x < 4$. **1**

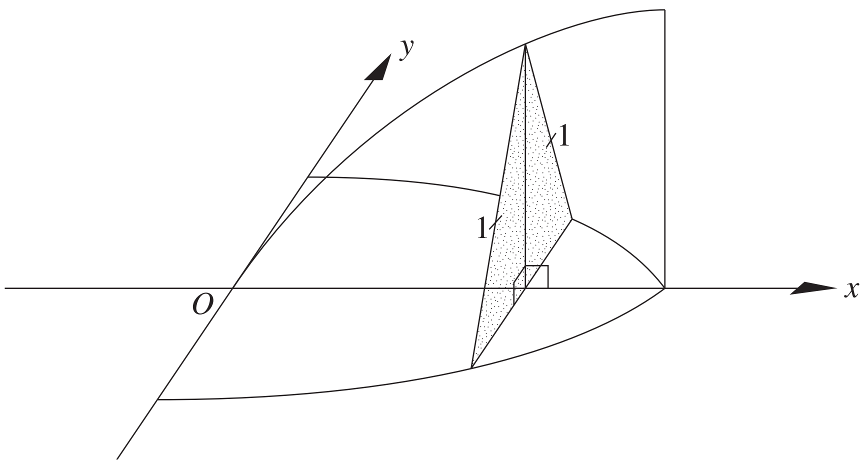
(ii) Find $\lim_{x \to 0} \dfrac{x}{\sin\frac{\pi}{2}x}$. **1**

(iii) Draw a one-third page sketch of the graph $y = \dfrac{x}{\sin\frac{\pi}{2}x}$ for $0 < x < 4$. **2**

(Do NOT calculate the coordinates of any turning points.)

(b) The base of a solid is formed by the area bounded by $y = \cos x$ and $y = -\cos x$ for $0 \le x \le \frac{\pi}{2}$. **3**

Vertical cross-sections of the solid taken parallel to the y-axis are in the shape of isosceles triangles with the equal sides of length 1 unit as shown in the diagram.



Find the volume of the solid.

Question 3 continues on page 5

(c) Use mathematical induction to prove that $(2n)! \geq 2^n (n!)^2$ for all positive integers n. **3**

(d) The equation $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ represents a hyperbola.

(i) Find the eccentricity e. **1**

(ii) Find the coordinates of the foci. **1**

(iii) State the equations of the asymptotes. **1**

(iv) Sketch the hyperbola. **1**

(v) For the general hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, describe the effect on the hyperbola as $e \to \infty$. **1**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet.

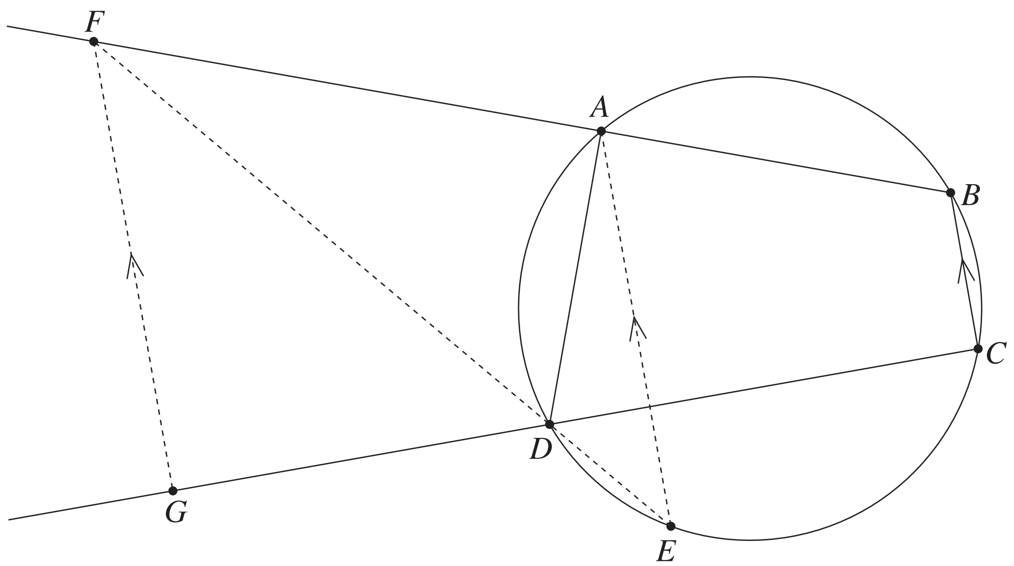
(a) Let a and b be real numbers with $a \neq b$. Let $z = x + iy$ be a complex number such that

$$|z-a|^2 - |z-b|^2 = 1.$$

(i) Prove that $x = \dfrac{a+b}{2} + \dfrac{1}{2(b-a)}$. **2**

(ii) Hence, describe the locus of all complex numbers z such that $|z-a|^2 - |z-b|^2 = 1$. **1**

(b) In the diagram, $ABCD$ is a cyclic quadrilateral. The point E lies on the circle through the points A, B, C and D such that $AE \parallel BC$. The line ED meets the line BA at the point F. The point G lies on the line CD such that $FG \parallel BC$.



Copy or trace the diagram into your writing booklet.

(i) Prove that $FADG$ is a cyclic quadrilateral. **2**

(ii) Explain why $\angle GFD = \angle AED$. **1**

(iii) Prove that GA is a tangent to the circle through the points A, B, C and D. **2**

Question 4 continues on page 7

(c) A mass is attached to a spring and moves in a resistive medium. The motion of the mass satisfies the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0,$$

where y is the displacement of the mass at time t.

(i) Show that, if $y = f(t)$ and $y = g(t)$ are both solutions to the differential equation and A and B are constants, then **2**

$$y = Af(t) + Bg(t)$$

is also a solution.

(ii) A solution of the differential equation is given by $y = e^{kt}$ for some values of k, where k is a constant. **2**

Show that the only possible values of k are $k = -1$ and $k = -2$.

(iii) A solution of the differential equation is **3**

$$y = Ae^{-2t} + Be^{-t}.$$

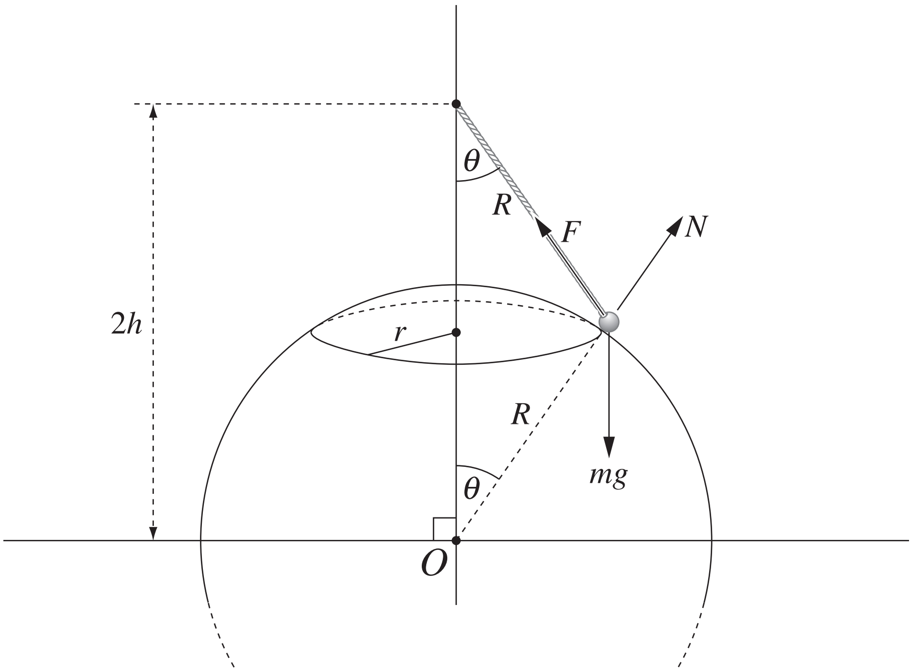
When $t = 0$, it is given that $y = 0$ and $\frac{dy}{dt} = 1$.

Find the values of A and B.

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) A small bead of mass m is attached to one end of a light string of length R. The other end of the string is fixed at height $2h$ above the centre of a sphere of radius R, as shown in the diagram. The bead moves in a circle of radius r on the surface of the sphere and has constant angular velocity $\omega > 0$. The string makes an angle of θ with the vertical.



Three forces act on the bead: the tension force F of the string, the normal reaction force N to the surface of the sphere, and the gravitational force mg.

(i) By resolving the forces horizontally and vertically on a diagram, show that **2**

$$F\sin\theta - N\sin\theta = m\omega^2 r$$

and

$$F\cos\theta + N\cos\theta = mg.$$

(ii) Show that **2**

$$N = \frac{1}{2}mg\sec\theta - \frac{1}{2}m\omega^2 r\,\mathrm{cosec}\,\theta.$$

(iii) Show that the bead remains in contact with the sphere if $\omega \le \sqrt{\frac{g}{h}}$. **2**

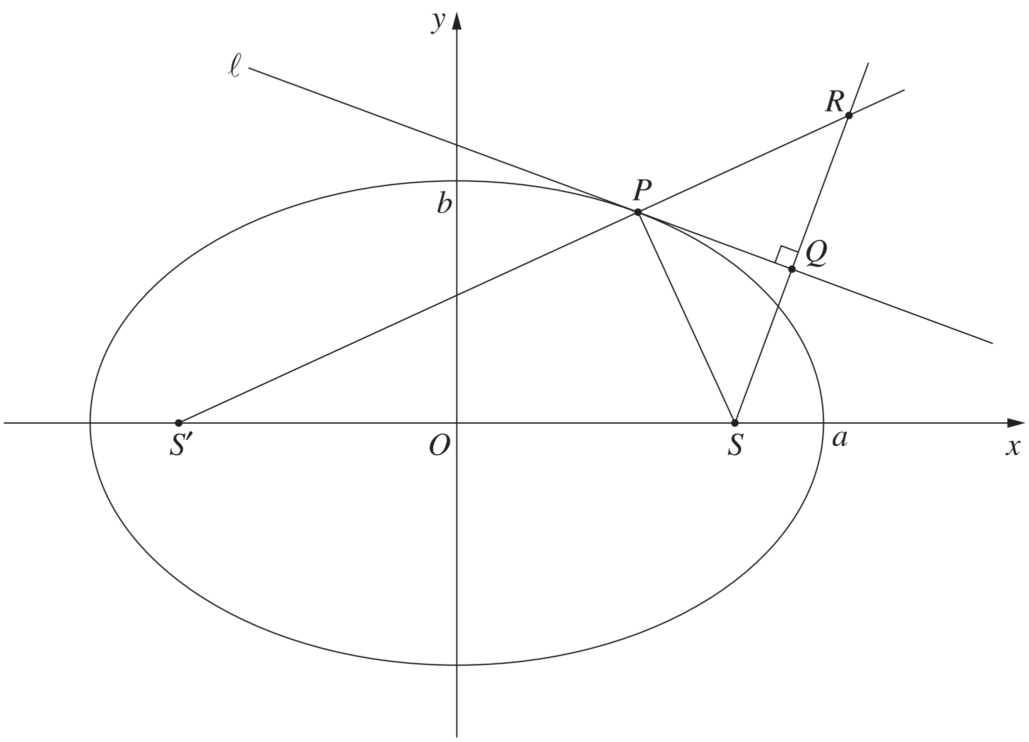
Question 5 continues on page 9

Question 5 (continued)

(b) If p, q and r are positive real numbers and $p + q \geq r$, prove that **3**

$$\frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r} \geq 0.$$

(c) The diagram shows the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$. The line ℓ is the tangent to the ellipse at the point P. The foci of the ellipse are S and S'. The perpendicular to ℓ through S meets ℓ at the point Q. The lines SQ and $S'P$ meet at the point R.



Copy or trace the diagram into your writing booklet.

(i) Use the reflection property of the ellipse at P to prove that $SQ = RQ$. **2**

(ii) Explain why $S'R = 2a$. **1**

(iii) Hence, or otherwise, prove that Q lies on the circle $x^2 + y^2 = a^2$. **3**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) Jac jumps out of an aeroplane and falls vertically. His velocity at time t after his parachute is opened is given by $v(t)$, where $v(0) = v_0$ and $v(t)$ is positive in the downwards direction. The magnitude of the resistive force provided by the parachute is kv^2, where k is a positive constant. Let m be Jac's mass and g the acceleration due to gravity. Jac's terminal velocity with the parachute open is v_T.

Jac's equation of motion with the parachute open is

$$m\frac{dv}{dt} = mg - kv^2. \quad \text{(Do NOT prove this.)}$$

(i) Explain why Jac's terminal velocity v_T is given by $\sqrt{\frac{mg}{k}}$. **1**

(ii) By integrating the equation of motion, show that t and v are related by the equation **3**

$$t = \frac{v_T}{2g}\ln\left[\frac{(v_T + v)(v_T - v_0)}{(v_T - v)(v_T + v_0)}\right].$$

(iii) Jac's friend Gil also jumps out of the aeroplane and falls vertically. Jac and Gil have the same mass and identical parachutes. **3**

Jac opens his parachute when his speed is $\frac{1}{3}v_T$. Gil opens her parachute when her speed is $3v_T$. Jac's speed increases and Gil's speed decreases, both towards v_T.

Show that in the time taken for Jac's speed to double, Gil's speed has halved.

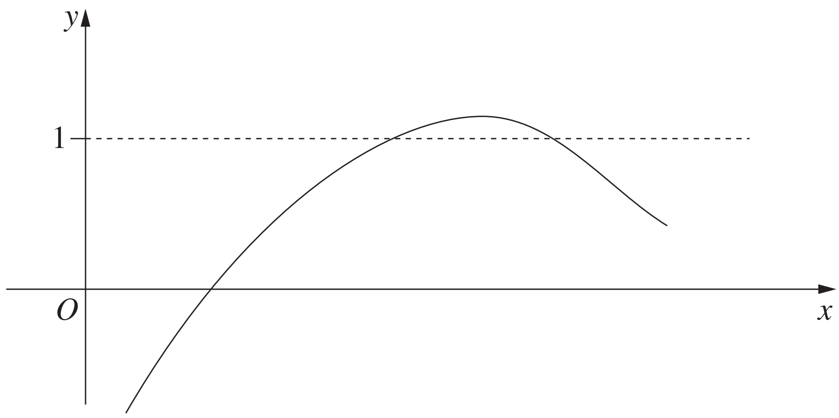
Question 6 continues on page 11

(b) Let $f(x)$ be a function with a continuous derivative.

(i) Prove that $y = (f(x))^3$ has a stationary point at $x = a$ if $f(a) = 0$ or $f'(a) = 0$. **2**

(ii) Without finding $f''(x)$, explain why $y = (f(x))^3$ has a horizontal point of inflexion at $x = a$ if $f(a) = 0$ and $f'(a) \neq 0$. **1**

(iii) The diagram shows the graph $y = f(x)$. **3**



Copy or trace the diagram into your writing booklet.

On the diagram in your writing booklet, sketch the graph $y = (f(x))^3$, clearly distinguishing it from the graph $y = f(x)$.

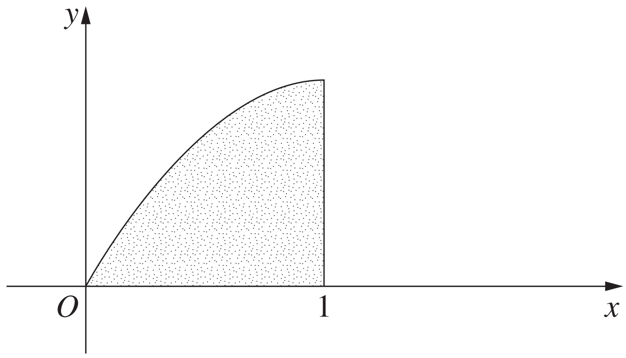
(c) On an Argand diagram, sketch the region described by the inequality **2**

$$\left|1 + \frac{1}{z}\right| \leq 1.$$

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of $f(x) = \dfrac{x}{1+x^2}$ for $0 \le x \le 1$. **4**



The area bounded by $y = f(x)$, the line $x = 1$ and the x-axis is rotated about the line $x = 1$ to form a solid.

Use the method of cylindrical shells to find the volume of the solid.

(b) Let $I = \displaystyle\int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\,dx$.

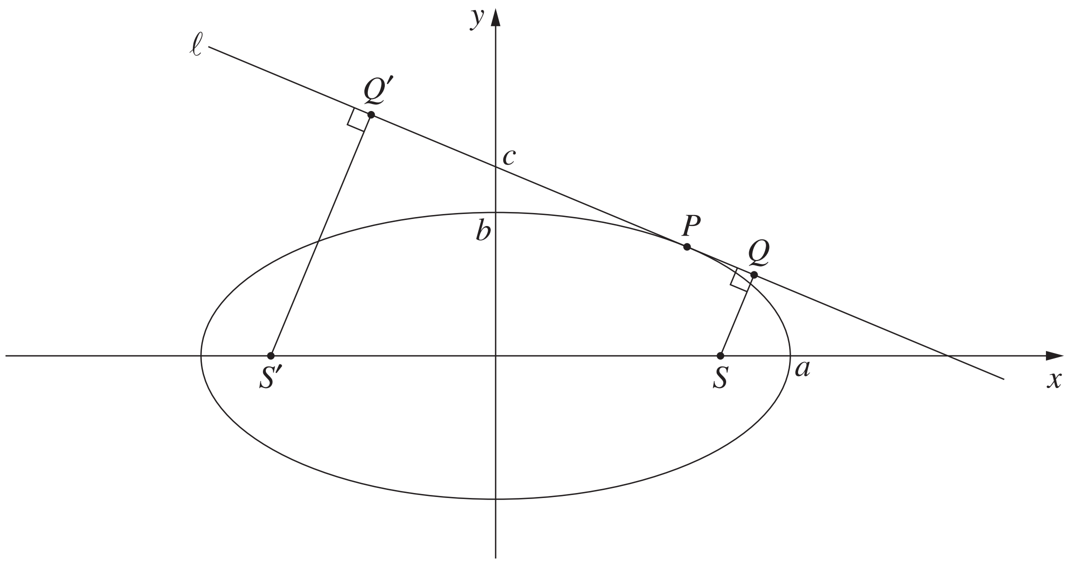
(i) Use the substitution $u = 4 - x$ to show that **2**

$$I = \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\,du\ .$$

(ii) Hence, find the value of I. **3**

Question 7 continues on page 13

(c) The diagram shows the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b$. Let e be the eccentricity of the ellipse.



The line ℓ is the tangent to the ellipse at the point P. The line ℓ has equation $y = mx + c$, where m is the slope and c is the y-intercept.

The points S and S' are the focal points of the ellipse, where S is on the positive x-axis. The perpendiculars to ℓ through S and S' intersect ℓ at Q and Q' respectively.

(i) By substituting the equation for ℓ into the equation for the ellipse, show that **3**

$$a^2m^2 + b^2 = c^2.$$

(ii) Show that the perpendicular distance from S to ℓ is given by **1**

$$QS = \frac{|mae + c|}{\sqrt{1 + m^2}}.$$

(iii) It is given that $Q'S' = \dfrac{|mae - c|}{\sqrt{1 + m^2}}$. **2**

Hence, prove that $QS \times Q'S' = b^2$.

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) For every integer $m \geq 0$ let **3**

$$I_m = \int_0^1 x^m \left(x^2 - 1\right)^5 dx .$$

Prove that for $m \geq 2$

$$I_m = \frac{m-1}{m+11} I_{m-2} .$$

(b) A bag contains seven balls numbered from 1 to 7. A ball is chosen at random and its number is noted. The ball is then returned to the bag. This is done a total of seven times.

(i) What is the probability that each ball is selected exactly once? **1**

(ii) What is the probability that at least one ball is not selected? **1**

(iii) What is the probability that exactly one of the balls is not selected? **2**

Question 8 continues on page 15

(c) Let β be a root of the complex monic polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 .$$

Let M be the maximum value of $|a_{n-1}|, |a_{n-2}|, \cdots, |a_0|$.

(i) Show that $|\beta|^n \le M\left(|\beta|^{n-1} + |\beta|^{n-2} + \cdots + |\beta| + 1\right)$. **2**

(ii) Hence, show that for any root β of $P(z)$ **3**

$$|\beta| < 1 + M .$$

(d) Let $S(x) = \sum_{k=0}^{n} c_k \left(x + \frac{1}{x}\right)^k$, where the real numbers c_k satisfy $|c_k| \le |c_n|$ **3**

for all $k < n$, and $c_n \neq 0$.

Using part (c), or otherwise, show that $S(x) = 0$ has no real solutions.

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$